A CHARACTERIZATION OF G-SPACES

BY

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ABSTRACT

Two theorems of Effros about G-spaces are proved without his hypothesis of separability.

Proposition 1 gives a proof, without the hypothesis of separability, of a lemma of Effros [1, Lemma 6.2] which describes how G-characters of a G-space sit in the dual space. This result has been obtained independently by Fakhoury [2, Théorème 6] with a different proof.

Theorem 1 uses this result to remove the hypothesis of separability from a Theorem of Effros [1, Theorem 6.3] which characterizes G-spaces amongst Lindenstrauss spaces in terms of the extreme points of the conjugate ball.

Our notation will follow that of [1]. V will always denote a real Lindenstrauss space and K the unit ball of V^* with the weak* topology. By a measure on K we will always mean a regular Borel measure. If μ is a measure on K then $\sigma\mu$ is the measure defined by $\sigma\mu(B) = \mu(-B)$ for Borel sets $B \subset K$. A family $\{\nu_i\}$ of measures on K is called pairwise singular if $\nu_i \wedge \nu_j = 0$ whenever $i \neq j$.

LEMMA 1. *Suppose V is a Lindenstrauss space, $\{\mu_i\}$ is a finite set of maximal probability measures on K such that the family $\{\mu_i\} \cup \{\sigma\mu_i\}$ is pairwise singular, and $\{\alpha_i\}$ is a finite set of real numbers with $|\alpha_i| \leqq 1$ for all i. Then for any $\varepsilon > 0$ there exists $h \in V$, $\|h\| \leqq 1$, such that for all i*

$$\mu_i(|h - \alpha_i| > \varepsilon) < \varepsilon.$$

PROOF. Choose $\delta = \varepsilon^2/6$ and choose compact sets $K_i \subset K$ such that $\mu_i(K_i) > 1 - \delta$ and the family $\{K_i\} \cup \{-K_i\}$ is pairwise disjoint. Choose d con-

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tinuous on K, $\|d\| = 1$, and $d(K_i) = \alpha_i$, $d(-K_i) = -\alpha_i$. Then $f(x) = \frac{1}{2}[d(x) - d(-x)]$ gives a function on K with the above properties of d, but which is also symmetric ($f(x) = -f(-x)$). Then

$$\mu_i(\bar{f}) = \mu_i(f) < \alpha_i + \delta$$

and

$$\sigma\mu_i(\bar{f}) < -\alpha_i + \delta$$

by maximality of μ_i and $\sigma\mu_i$ [5, Proposition 4.2]. Choose g concave and continuous on K so that $-1 \leqq \bar{f} \leqq g \leqq 1$ and

$$\mu_i(g) < \alpha_i + \delta$$

$$\sigma\mu_i(g) < -\alpha_i + \delta \quad [4, \text{Chp. II, T36}].$$

Then for $x \in E(K)$, $g(x) + g(-x) \geqq f(x) + f(-x) = 0$. Choose $h \in V$ such that $h \leqq g$ on K [3, Theorem 2.1].

Let $N_i = \{g > \alpha_i + \varepsilon\}$ and suppose $\mu_i(N_i) = \gamma \geqq \varepsilon/2$. Then if $\beta = \mu_i(K_i^c \cup N_i)$,

$$\begin{aligned}
\mu_i(g) &\geqq \int_{K_i \cap N_i^c} g d\mu_i + \int_{N_i} g d\mu_i - \delta \\
&\geqq \alpha_i(1-\beta) + (\alpha_i + \varepsilon)\gamma - \delta \\
&= \alpha_i(1 + \gamma - \beta) + \varepsilon\gamma - \delta \\
&\geqq \alpha_i - 2\delta + \varepsilon^2/2 = \alpha_i + \delta \text{ (note } \gamma - \beta \geqq -\delta),
\end{aligned}$$

a contradiction.

So $\mu_i(N_i) < \varepsilon/2$. Since $h \leqq g$

$$\mu_i(\{h > \alpha_i + \varepsilon\}) \leqq \mu_i(N_i) < \varepsilon/2.$$

On the other hand

$$\mu_i(\{h < \alpha_i - \varepsilon\}) = \mu_i(\{-h > -\alpha_i + \varepsilon\}) =$$

$$\sigma\mu_i(\{h > -\alpha_i + \varepsilon\}) \leqq \sigma\mu_i(\{g > -\alpha_i + \varepsilon\}) < \varepsilon/2.$$

The last inequality follows by an argument similar to the above using the fact that $\sigma\mu_i(g) < -\alpha_i + \delta$.

Proposition 1. *If V is a G-space then the G-characters are the elements in $RE(K)$.*

Proof. The proof of [1, Lemma 6.2] shows that any $x \in RE(K)$ is a G-character. Conversely suppose x is a G-character, $\|x\| = 1$ and $x \notin E(K)$. We will be finished if we obtain a contradiction. Choose a maximal probability measure μ on K with barycentre x. Since μ is maximal and $x \notin E(K)$, μ is not a point mass (one way to prove this is from [5, Proposition 4.2]), so there are maximal probability measures μ_1 and μ_2 on K and $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$, so that $\mu = \alpha_1 \mu_1 + \alpha_2 \mu_2$ and $\mu_1 \wedge \mu_2 = 0$. Also $\mu \wedge \sigma\mu = 0$. Indeed $\omega = \mu \wedge \sigma\mu$ is symmetric ($\omega = \sigma\omega$), so has barycentre 0. Hence $\mu - \omega$ has barycentre x. Since $\|x\| = 1$ and $\mu - \omega$ lives on K, we have $\|\mu - \omega\| \geqq 1$; since $0 \leqq \omega \leqq \mu$ we deduce $\omega = 0$. So $\{\mu_1, \mu_2, \sigma\mu_1, \sigma\mu_2\}$ is pairwise singular.

Using Lemma 1 with $\varepsilon = \alpha_1\alpha_2/14$, choose $f, g \in V$, $\|f\| = \|g\| = 1$, so that

$$\mu_1(|f - \alpha_2| > \varepsilon) < \varepsilon \qquad \mu_1(|g - 1| > \varepsilon) < \varepsilon$$

$$\mu_2(|f + \alpha_1| > \varepsilon) < \varepsilon \qquad \mu_2(|g| > \varepsilon) < \varepsilon.$$

Let $h = f \curlywedge g$ and define k on K by the formula

$$k(y) = f(y) \curlywedge g(y) \quad \text{for } y \in K.$$

Then $h \in A$ and k is continuous, so the set $\{h = k\}$ is closed and contains $E(K)$, so supports the maximal measures μ_1 and μ_2. Hence

$$\begin{aligned} h(x) &= \mu(h) = \alpha_1\mu_1(h) + \alpha_2\mu_2(h) \\ &= \alpha_1\mu_1(k) + \alpha_2\mu_2(k) \\ &> \alpha_1[(\alpha_2 - \varepsilon)(1 - 2\varepsilon) - 2\varepsilon] + \alpha_2[(-\varepsilon)(1 - \varepsilon) - \varepsilon] \\ &= \alpha_1\alpha_2(1 - 2\varepsilon) - \alpha_1(3\varepsilon - 2\varepsilon^2) - \alpha_2(2\varepsilon - \varepsilon^2) \\ &> \frac{\alpha_1\alpha_2}{2} - 3\varepsilon - 2\varepsilon = 2\varepsilon. \end{aligned}$$

The first inequality is obtained by noting that if $f(y) > \alpha_2 - \varepsilon$ and $g(y) > 1 - \varepsilon$ then $k(y) > \alpha_2 - \varepsilon$ and this happens except on a set of μ_1-measure $< 2\varepsilon$; similarly $g(y) > -\varepsilon$ and therefore $k(y) > -\varepsilon$ except on a set of μ_2-measure $< \varepsilon$; finally $k(y) \geqq -1$ on the exceptional sets.

On the other hand since x is a G-character

$$h(x) = f(x) \curlywedge g(x) = \mu(f) \curlywedge \mu(g).$$

Now

$$\begin{aligned} |\mu(f)| &= |\alpha_1\mu_1(f) - \alpha_1\alpha_2 + \alpha_2\mu_2(f) + \alpha_1\alpha_2| \\ &\leqq \alpha_1|\mu_1(f) - \alpha_2| + \alpha_2|\mu_2(f) + \alpha_1| \\ &\leqq \alpha_1 2\varepsilon + \alpha_2 2\varepsilon = 2\varepsilon. \end{aligned}$$

The last estimate is an easy consequence of the assumptions on f. So

$$|h(x)| = |\mu(f) \curlywedge \mu(g)| \leqq 2\varepsilon,$$

a contradiction.

THEOREM 1. *If V is a Lindenstrauss space then the following are equivalent.*

(1) *The structure topology on $E_\sigma(K)$ is Hausdorff.*

(2) *The closure Z of $E(K)$ is contained in $[0,1]E(K)$.*

(3) *V is a G-space.*

PROOF. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is proved in [1, Theorem 6.3].

(3) $\Rightarrow$ (2). From Lemma 1 $[0,1]E(K) = \{G\text{-characters}\} \cap K$ and is therefore weak* closed.

(2) $\Rightarrow$ (1). Suppose e_1 and e_2 are in $E(K)$ and give different points in $E_\sigma(K)$; so $e_1 \neq \pm e_2$. Choose f and g in V such that

$$f(e_1) = f(e_2) = 1 = g(e_1) = -g(e_2).$$

Let D_1 (resp. D_2) be the subset of $[0,1]E(K)$ on which f and g have the same (resp. different) sign. Precisely

$$D_1 = \{p \in [0,1]E(K) : f(p) > 0 \Rightarrow g(p) \geqq 0 \text{ and } f(p) < 0 \Rightarrow g(p) \leqq 0\},$$

$$D_2 = \{p \in [0,1]E(K) : f(p) > 0 \Rightarrow g(p) \leqq 0 \text{ and } f(p) < 0 \Rightarrow g(p) \geqq 0\}.$$

The D_i are clearly symmetrically dilated and are weak* compact since $[0,1]E(K)$ is weak* closed. So by [1, Theorem 5.8] $D_i \cap E(K)$ is structurally closed for $i = 1,2$. Clearly $e_1 \notin D_2$, $e_2 \notin D_1$ and $E(K) \subset D_1 \cup D_2$. Thus the sets $E(K) - D_i$ are symmetric and structurally open and separate e_1 and e_2.

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